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Simplicial Schemes*

BOJAN MOHAR

Department of Mathematics, University E.K. of Ljubljana
Jadranska 19, 61111 Ljubljana, Yugoslavia

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A simplicial scheme is a certain structure which can be defined on graphs. The purpose of this concept is a graph-theoretical description of simplicial complexes. It is shown that graphs and simplicial schemes give rise precisely to simplicial pseudocomplexes which are pure and in which the open star of every simplex is strongly connected and every simplex of codimension one is contained in at most two top-dimensional simplices. A characterization is given for a complex arising from a graph and a simplicial scheme to be orientable. Finally, a relation between graph maps and nondegenerate simplicial maps of associated complexes is considered. © 1987 Academic Press, Inc.

1. INTRODUCTION

Suppose that we are given an n-dimensional simplicial complex K. Usually we describe its structure by the incidence relations between the simplices of K. Such a description is tedious and space consuming in practical applications and is not very useful in many instances. For a large class of (pseudo) triangulations, the so called relative cycles, which include all manifolds with boundary, we introduce a new way of describing their combinatorial structure. The description is purely graph-theoretical and by using its properties we obtain some nontrivial combinatorial results. An important fact is that the covering projections between graphs correspond to branched covering projections between complexes described by the graphs. Hence we may use any graph covering tool, e.g., *voltage graphs* [7, 8], to describe branched coverings (cf. [12, 14]).

At the time of writing, some papers which include a description of simplicial complexes similar to our own became known to us. An approach similar to ours was firstly given by Ferri, Gagliardi, Pezzana, *et al.* [1, 3, 4, 5, 6, 17]. They use edge-colored graphs to encode certain simplicial complexes. Each edge-colored graph can be viewed as a simplicial scheme

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on the same graph, and the mentioned encoding of simplicial complexes is the same as ours with this particular simplicial scheme. However, the main goal of Pezzana group is topological. For a given *PL*-manifold they wish to obtain a pseudotriangulation which will be as small as possible. This leads to the concepts of a *contracted triangulation*, i.e., one such that its set of 0-simplices has cardinality $n+1$ where n is the dimension of the manifold which is considered. Pezzana [17] has shown that every closed *PL*-manifold admits a contracted triangulation, and this can be represented by an edge-colored graph. Such a description of closed *PL*-manifolds is useful for some topological purposes, e.g., computing the fundamental group, classifying *PL*-manifolds, computing homology groups, etc. In another paper, written by Lins [11], only the case in dimension two is considered. A related concept was also introduced by Vince [18, 19, 20], where it is called a *combinatorial map*. The setting of these papers is more combinatorial, but the description coincides with that of Ferri, Gagliardi, Pezzana, *et al.*; edge-colored graphs are used again. The use of combinatorial maps was very successful in two ways. First, highly regular combinatorial maps (e.g., tilings) can be handled using purely algebraic methods [19], and second, some work has been done for introducing new techniques from a totally combinatorial viewpoint, e.g., edge-shellability [20].

When comparing with the edge-colored graphs, the more general simplicial scheme approach has many disadvantages. Simplicial schemes are more general than edge-colored graphs, in that they can encode any simplicial complex which is "nice" enough, while edge-colored graphs provide encoding only for barycentric subdivisions of "nice" cell complexes. The price paid for the greater generality is a much more complicated mathematical aparatus.

We expect, however, that simplicial schemes will be successfully applied in some other directions than the applications of edge-colored graphs. With simplicial schemes we can encode each relative cycle (these are, roughly speaking, pseudomanifolds of dimension n which are locally strongly connected). The class of relative cycles contains two important families of complexes: all triangulated manifolds (with boundary) and all simplicial polytopes. Even though not explicitly stated in the paper, it is clear that the same encoding will be good for simple polytopes, the duals of simplicial polytopes. These are difficult to visualize in higher dimensions. However, the use of simplicial schemes makes their visualization as easy as possible.

Another advantage of simplicial schemes is in describing nonsingular simplicial maps and , in particular, branched covering projections between relative cycles (see e.g., [14]).

Given a graph and a set of "local mappings," it is difficult to check if these data have all the properties needed to be a simplicial scheme. Therefore, the main applications of simplicial schemes will be when they are obtained in such a way that the construction implies the required properties. This is true, for example, when the simplicial scheme is obtained from a given relative cycle, or when it is obtained as the lift of another simplicial scheme (see Sect. 4). At the end of Section 4 we give an example how simplicial schemes can be used in proving that under some conditions the dual graph of a relative cycle is bipartite.

In Section 2 we give some basic definitions of graph theory and of combinatorial topology. Section 3 introduces the concept of simplicial scheme. It is shown how graphs and simplicial schemes relate to (simplicial) pseudocomplexes, and we obtain a simple characterization of those pseudocomplexes which can be represented by the dual graph and a simplicial scheme on it. These pseudocomplexes are precisely the relative cycles. Finally we show how the orientability of the pseudocomplex associated with a graph and a simplicial scheme can be seen from the graph itself, and we also present the relation between the join of pseudocomplexes and the Cartesian product of dual graphs. We note that the results about the orientability and the join were also obtained in the setting of edge-colored graphs [4, 11, 18].

In Section 4 we investigate the relation between the graph maps and the corresponding simplicial maps. As noted before, the simplicial branched coverings between relative cycles are in a natural bijective correspondence with graph covering projections between the dual graphs. This simple fact has a number of interesting consequences which the author intends to publish in a subsequent paper on simplicial schemes [14]. At the end of Section 4 it is shown that the colorings of the vertices of relative cycles can be expressed in terms of simplicial schemes.

We note that a somewhat restricted theory of simplicial schemes (only for pseudocomplexes without boundary) was presented at the Fourth Yugoslav meeting on graph theory [13].

2. Basic Definitions

For our purposes, a graph is what is sometimes called a "pseudograph," i.e., loops and multiple edges are allowed. To be more precise, we shall distinguish between two kinds of loops: proper loops and half-edges. When counting the degree of a vertex in a graph which is drawn on a surface, one counts two for each proper loop, since in a small neighbourhood of the vertex a loop looks like two lines issuing out of the vertex. Another kind of loop which we shall use is called a "half-edge" and can be interpreted as an edge having only one end. Thus we define: a *graph* G is a quadruple $G = (V, S, r, i)$ in which V and S are finite sets, r is an involution of S, and i is a function $S \to V$. The elements of V are the *vertices* and the elements of

S are the *arcs* of the graph. For each $e \in S$, $r(e)$ is the *opposite arc* of e, $i(e)$ is the *initial vertex*, and $t(e) := ir(e)$ is the *terminal vertex* of e. If e is an arc of G such that $i(e) = t(e)$, then e is called a *loop*. The loops are divided into two classes. If $r(e) \neq e$ then the loop e is a *proper loop*, otherwise it is called a *half-edge*. Finally, an *edge* of G is an orbit of r, i.e., a set $\{e, r(e)\}$ for some $e \in S$. The one-element edges are conveniently termed *half-edges*. In the sequel we consider only graphs without proper loops.

If we refer to a graph G without an explicit note on V, S, r, and i, we shall denote these quantities by $V(G)$, $S(G)$, r_G, and i_G, respectively. We shall suppress the index "G" if there is no confusion as to which graph is considered. The same applies to the function $t_G := i_G r_G$.

For each $v \in V(G)$ we define the *star* of v, denoted by $\mathrm{st}(v, G)$, as the set of the arcs which have v as the initial vertex, $\mathrm{st}(v, G) = \{e \in S(G); i(e) = v\}$. The *degree* of v, denoted by $\deg_G(v)$ or simply by $\deg(v)$, is equal to the cardinality of the star of v. G is said to be *r-regular* if $\deg(v) = r$ for every vertex v of G.

For our purposes, a *graph map* of a graph G into a graph H is a pair $p = (p_V, p_S)$, where $p_V\colon V(G) \to V(H)$ and $p_S\colon S(G) \to S(H)$ are maps such that

(1) $p_V i_G = i_H p_S$, or, equivalently, for each $v \in V(G)$, $p_S(\mathrm{st}(v, G))$ is a subset of $\mathrm{st}(p_V(v), H)$, and

(2) for each $e \in S(G)$, either $p_S r_G(e) = r_H p_S(e)$ or $p_S r_G(e) = p_S(e)$.

For convenience we write $p(v)$ and $p(e)$ instead of $p_V(v)$ and $p_S(e)$, respectively. Similar we use the notation $p\colon G \to H$ to denote that p is a graph map from G to H. Usually in graph theory only graph maps which map edges to edges are considered. Note that by our definition it may happen that a graph map $p\colon G \to H$ maps a half-edge onto an arc which is not a half-edge, or even worse, p may map a proper edge onto an arc which is not a half-edge, i.e., $e \neq r(e)$ and $p(e) = p(r(e)) \neq r(p(e))$. Later we shall see why this peculiarity is necessary.

The arcs of G can be classified into two classes with respect to a given graph map p. An arc $e \in S(G)$ is said to be *singular* if $e \neq r(e)$ and $p(e) = pr(e)$. Otherwise e is a *nonsingular* arc. Note that, if e is singular, then also its opposite arc is singular. This means that also the edges of G can be classified as singular and nonsingular with respect to p. A half-edge is always nonsingular. A proper edge is singular if and only if it is not mapped onto a proper edge. Geometrically this means a folding of the edge.

Let $p\colon G \to H$ be a graph map. If for each $v \in V(G)$, $p(\mathrm{st}(v, G)) = \mathrm{st}(p(v), H)$, then p is said to be an *s-map*. A convenient property of s-maps is the following. If an s-map $p\colon G \to H$ maps edges onto edges and H is connected, then p is onto. Graphs and s-maps obiously form a category. The full subcategory containing r-regular graphs is denoted by GPH_r. Since an s-map, which has its inverse in GPH_r, maps edges to edges, it is easy to see that isomorphisms in GPH_r are the usual graph isomorphsims.

A walk in a graph G is a sequence of arcs $W = e_1 e_2 ... e_d$ such that $t(e_j) = i(e_{j+1})$ for $j = 1, 2, ..., d-1$. The walk W is *closed* if $t(e_d) = i(e_1)$.

We proceed with defining some concepts of topology and combinatorial topology. We assume all standard definitions and shall limit ourselves to defining only less known terms and those which may cause confusion. A *topological simplex* of dimension n (also *n-simplex*) is a topological space D together with a homeomorphism $\Delta^n \to D$ where Δ^n is the standard n-simplex. The given homeomorphism determines the facial and the linear structure of D. By this definition, a face of a simplex has the linear structure which is induced by the linear structure of the simplex.

A collection K of topological simplices is called a *pseudocomplex* if the following conditions are satisfied:

(1) If $A \in K$ and B is a face of A, then $B \in K$

(2) If $A, B \in K$, then the intersection $A \cap B$ is a union (possibly empty) of simplices of K

(3) If $|K|$ is the union of all simplices of K, then $|K|$ is also equal to the disjoint union of interiors of simplices of K.

Rougly speaking, a pseudocomplex K is a collection of topological simplices such that the intersection of any two of them is a union of simplices of K. This notion is a special case of a more general concept: a pseudocomplex is a particular cell complex [2] whose k-cells considered will all their faces are isomorphic to k-simplices. The notion of a pseudocomplex was introduced in [9] in order to simplify the calculations of homology groups, since a pseudocomplex K may happen to have considerably fewer simplices than any triangulation of $|K|$. Note that the first barycentric subdivision of a pseudocomplex is a simplicial complex, hence the underlying topological space $|K|$ of a pseudocomplex K is a polyhedron.

In the obvious manner we carry some definitions concerning simplicial complexes to pseudocomplexes. This includes the *dimension* of a pseudocomplex, the *link*, the *star*, and the *open star* of a simplex, the concepts of a *simplicial map*, *nondegenerate map*, etc. If $f: K \to L$ is a simplicial map between pseudocomplexes K and L, it induces a continuous map $|f|: |K| \to |L|$ which is determined as follows. If $x \in V(K)$ ($V(K)$ is the set of 0-simplices of K), then let $|f|(x) := f(x) \in |L|$. Next extend $|f|$ to $|K|$ by requiring that it is linear in the interior of the simplices of K.

A collection K of simplices is *pure* of dimension n if every simplex in K is contained in some n-simplex of K. K is *strongly connected* if for any two top-dimensional simplices A and B of K there exists a sequence of top-dimensional simplices $A = P_1, P_2, \ldots, P_{k-1}, P_k = B$, such that P_j and P_{j+1}, $j = 1, 2, \ldots, k-1$, have a common face of codimension one. Suppose that

(1) K is pure,

(2) for every simplex A of K, the open star of A is strongly connected, and

(3) every $(n-1)$-simplex of K is contained in at most two n-simplices of K.

Then K is said to be a *relative n-cycle*. The *boundary* of K, denoted by $\mathrm{bd}(K)$, is the subcomplex of K induced by $(n-1)$-simplices which are contained in exactly one n-simplex of K. If $\mathrm{bd}(K) \neq \varnothing$ then K is a (geometric) *n-cycle* as it is also called by Latour [10]. We note that (1), (2), and (3) are in fact topological properties of the underlying polyhedron, i.e., if K and L are pseudocomplexes with homeomorphic underlying topological spaces and if K is a relative n-cycle, then L is a relative n-cycle, too. Each relative cycle is a pseudomanifold.

Some examples of relative 2-cycles are given in Fig. 1. The arrows on edges indicate instructions for gluing edges together. The pseudocomplexes of Figures 1a, 1b, and 1d are not simplicial complexes.

Let K be an n-dimensional pseudocomplex in which every $(n-1)$-simplex is contained in at most two n-simplices. We define the *dual graph* G of K as follows. The vertex-set of G is the set of n-simplices of K, and $S(G)$ is the set of pairs (A, B) where A is an n-simplex of K and B is an $(n-1)$-face of A. For $e = (A, B)$ define $i(e) := A$ and $r(e) := (A', B)$ where $A' = A$ if $B \in \mathrm{bd}(K)$ (the only n-simplex which contains B is A) and otherwise A' is the n-simplex which contains B and is different from A. Note that the graph is $(n+1)$-regular and its connected components correspond to strongly connected components of K. For each boundary $(n-1)$-simplex we obtain a half-edge. If K has no boundary, then G has no half-edges and the

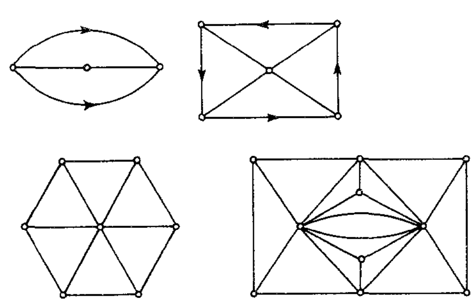

FIGURE 1

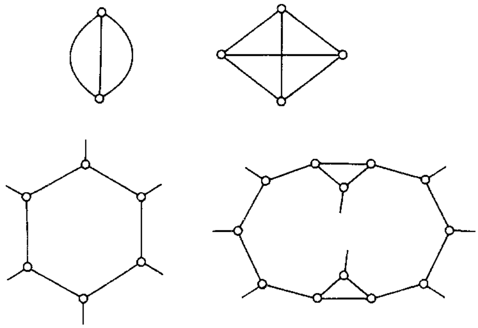

FIGURE 2

definition agrees with the usual one. The dual graph with deleted half-edges is just the 1-skeleton of the dual of K. Two arbitrary vertices of G are joined by one edge for each common $(n-1)$-face of the corresponding two n-simplices. The dual graphs of the pseudocomplexes of Fig. 1 are shown in Fig. 2.

3. SIMPLICIAL SCHEMES

Let G be a regular graph. A *presimplicial scheme* g on G is a function which assigns to every arc $e \in S(G)$ a bijective map $g(e)\colon \mathrm{st}(i(e), G) \to \mathrm{st}(t(e), G)$ with the following properties:

(SS1) for each arc $e \in S(G)$, $g(e)\, e = r(e)$, and

(SS2) for each arc $e \in S(G)$, $g(r(e)) = (g(e))^{-1}$.

Note that we write $g(e) f$ instead of $g(e)(f)$.

Let $W = f_1 f_2 \cdots f_d$ be a walk in G. Denote by $g(W)$ the composition $g(f_d) \circ g(f_{d-1}) \circ \cdots \circ g(f_1)$. An arc $e \in \mathrm{st}(i(f_1), G)$ is said to *avoid the walk* W (w.r.t. the presimplicial scheme g) if for $j = 0, 1, \ldots, d-1$,

$$g(f_1 f_2 \cdots f_j)\, e \neq f_{j+1}. \tag{3.1}$$

Note that the above condition for $j = 0$ yields that $e \neq f_1$.

A presimplicial scheme g is a *simplicial scheme* if, in addition to (SS1) and (SS2), also the following condition is satisfied:

(SS3) for each closed walk $W = f_1 f_2 \cdots f_d$ and each arc $e \in \mathrm{st}(i(f_1), G)$ which avoids W, $g(W)\, e = e$.

If g is a simplicial scheme and f is a loop, then the condition (SS3) for $W = f$ implies that

$$g(f) = id_{\mathrm{st}(i(f), G)}. \tag{3.2}$$

Simplicial schemes and pseudocomplexes are related by the following construction. Let G be an $(n+1)$-regular graph, $G \in GPH_{n+1}$, and let g be a simplicial scheme on G. Then we construct an n-dimensional pseudocomplex $K = K(G, g)$ such that G is isomorphic with the dual graph of K. For each vertex $v \in V(G)$ take an n-simplex A_v and choose a bijective correspondence between the arcs in $\mathrm{st}(v, G)$ and the $(n-1)$-faces of A_v. For each arc e of $\mathrm{st}(v, G)$ denote by face(e) the $(n-1)$-face of A_v corresponding to e, and let vert(e) be the vertex of A_v opposite face(e) (thus $A_v = \mathrm{vert}(e) * \mathrm{face}(e)$, where $*$ means join). We are now ready to define the pseudocomplex $K(G, g)$. It has n-simplices A_v, $v \in V(G)$, and if $f \in \mathrm{st}(v, G)$ then identify $\mathrm{face}(f) \subseteq A_v$ and $\mathrm{face}(r(f)) \subseteq A_{t(f)}$ so that for each arc $e \in \mathrm{st}(v, G) - \{f\}$ the vertices vert(e) and $\mathrm{vert}(g(f)\, e)$ are identified.

It is easy to see that a vertex α of A_v is identified with a vertex β of A_u (by successive identifications of $(n-1)$-simplices) if and only if in G there exists a walk W from v to u such that the arc e of $\mathrm{st}(v, G)$ with $\mathrm{vert}(e) = \alpha$ avoids W and $\beta = \mathrm{vert}(g(W)\, e)$. This shows why we need the condition (SS3). Suppose that there is a closed walk W and there is an arc e which avoid W but $g(W) \neq e$. The the distinct vertices vert(e) and $\mathrm{vert}(g(W)\, e)$, both belonging to the same n-simplex, will be identified, and the corresponding n-simplex will be deformed to something that is not a simplex. We remark that a theory of complexes built from such deformed simplices can also be developed.

Similarly as above, the n-simplices A_v and A_u intersect in a k-simplex if and only if there is a walk W from v to u such that there are arcs $e_0, e_1, \ldots, e_k$ of $\mathrm{st}(v, G)$ which avoid W. This basic property will be used in the sequel and will be of great importance when condidering the properties of pseudocomplexes obtained from graphs and simplicial schemes.

3.1. PROPOSITION. *Let X and Y be n-simplices of $K(G, g)$. If X and Y have a common k-face, then*

(1) *there exists a walk $W = f_1 f_2 \cdots f_d$ in G such that $X = A_{i(f_1)}$, and $Y = A_{t(f_d)}$, and*

(2) *there exist* arcs $e_0, e_1, \ldots, e_k$ *in* $\mathrm{st}(i(f_1), G)$ *which all avoid W.*

Conversely, if (1) *and* (2) *are satisfied then X and Y have a common k-face with vertices* $\mathrm{vert}(e_0), \mathrm{vert}(e_1), \ldots, \mathrm{vert}(e_k)$.

Let G be a regular graph and g a simplicial scheme on G. Let H be a regular subgraph of G. If for each $e \in S(H)$, $f(e)(\mathrm{st}(i(e), H)) = \mathrm{st}(t(e), H)$, then there is a simplicial scheme h on H such that for each $e \in S(H)$ and each $f \in \mathrm{st}(i(e), H)$, $h(e) f = g(e) f$. We say that h is the *restriction* of g to H, and we denote this by $h = g \mid H$.

3.2. PROPOSITION. *Let G be an $(n+1)$-regular graph and g a simplicial* *scheme on G. Suppose, moreover, that $K(G, g)$ is a simplicial complex. If H is a connected k-regular subgraph of $G(k \leq n)$ such that g can be restricted to H, then there is an $(n-k)$-simplex A of $K(G, g)$ such that* $\operatorname{link}(A, K(G, g)) \approx K(H, g \mid H)$.

Conversely, if A is an $(n-k)$-simplex of $K(G, g)$ then the subgraph H of G which contains precisely the arcs of G, associated with the $(n-1)$-simplices containing A, is k-regular and connected, and g can be restricted to H.

Proof. Let v be a vertex of H, and let $e_0, e_1, \ldots, e_{n-k}$ be the arcs of $\operatorname{st}(v, G)$ which do not belong to H. Take the $(n-k)$-face of the n-simplex A which contains vertices $\operatorname{vert}(e_0)$, $\operatorname{vert}(e_1), \ldots, \operatorname{vert}(e_{n-k})$. Let W be any walk in H which starts at v and ends, say, at u. Since g can be restricted to H, the arcs $e_0, e_1, \ldots, e_{n-k}$ avoid W. By Proposition 3.1 the n-simplex A_u also contains A. This shows that the vertices of H correspond to n-simplices of $K(G, g)$ which contain A, and the arcs of H correspond to $(n-1)$-simplices which contain A. This correspondence can be transfered to the correspondence of top-dimensional and codimension-one simplices of $\operatorname{link}(A, K)$ with the elements of H, and this naturally yields an isomorphism between $\operatorname{link}(A, K(G, g))$ and $K(H, g \mid H)$.

The proof of the second part of the proposition is similar and we omit it. ∎

We note that a similar property also holds for pseudocomplexes which are no simplicial complexes. Also in this case H is isomorphic with the graph corresponding to the open star of A; but instead of the link of A one should define the so called *disjoint link* which is, rougly speaking, the absolute "boundary" of the open star.

Proposition 3.2 enables us to describe the codimension-two simplices in $K(G, g)$. We state two corollaries whose easy proofs are left to the reader.

3.3. COROLLARY. *To each $(n-2)$-simplex A of $K(G, g)$ corresponds a connected 2-regular subgraph W of G of order r where r is the number of n-simplices which contain A. If A is an interior simplex then W is isomorphic to the cycle C_r of length r. If A is a boundary simplex then W is isomorphic to the path P_r on r vertices which has a half-edge attached at each end.*

3.4. COROLLARY. *Let $W = f_1 f_2 \cdots f_d$ be a closed walk in G, and let $f_0 := f_d$ and $f_{d+1} := f_1$. If*

$$g(f_j)\, r(f_{j-1}) = f_{j+1}, \qquad j = 1, 2, \ldots, d, \tag{3.3}$$

then W corresponds to the subgraph of an $(n-2)$-simplex of $K(G, g)$.

Conversely, ifA is an interior simplex of codimension two in $K(G, g)$, then its subgraph C_r can be represented by a walk of length $d = r$ for which (3.3) *holds. The subgraph P_r of a boundary $(n-2)$-simplex can be represented by a closed walk W of length $d=2r$ for which* (3.3) *holds. This walk contains each arc of P_r.*

3.5. EXAMPLE. Let H_m be an m-regular graph without half-edges on two vertices, say x and y. Note that H_m is determined up to isomorphism. Denote by $e_1, e_2, \ldots, e_m$ the arcs of H_m which have the initial vertex x. Define a simplicial scheme h_m on H_m by $h_m(e_j)\, e_k := r(e_k)$, $1 \leqq j \leqq m$, $1 \leqq k \leqq m$. It is easily verified that h_m is the only possible simplicial scheme on H_m. $K(H_{n+1}, h_{n+1})$ is a pseudotriangulation of the n-sphere. It consists of two n-simplices which share all $(n-1)$-faces.

3.6. EXAMPLE. Let W_m be the m-regular graph consisting of one vertex and m half-edges. By (3.2), W_m admits only one simplicial scheme, say w_m. Clearly, $K(W_{n+1}, w_{n+1})$ is isomorphic to the n-simplex Δ^n.

Suppose that an $(n+1)$-regular graph G can be edge-colored by $n+1$ colors in such a way that any two indicident edges receive different colors. A particular edge-coloring induces an arc-coloring: the arcs r, $r(e)$ are colored the same as the edge $\{e, r(e)\}$. For each $e \in S(G)$ and each $f \in \mathrm{st}(i(e), G)$, let $g(e)f$ be the arc of $\mathrm{st}(t(e), G)$ which has the same color as f. Obviously, g is a simplicial scheme, and we say that it is *induced by the edge-coloring*. This relation between edge-colorings and simplicial schemes was mentioned already in the introduction. By this, our theory is closely related with the work of Pezzana group [3, 4, 5, 6, 17], Lins [11], and Vince [18, 19, 20].

3.7. EXAMPLE. Take the complete graph K_4 on four vertices, and let g be the simplicial scheme which is induced by the edge-coloring of K_4. $K(K_4, g)$ has 4 triangles and 6 edges. The number of vertices will be equal to the number of 2-colored cycles of K_4 (cf. Corollary 3.3), which is easily seen to be 3. By Euler's formula, $K(K_4, g)$ has Euler characteristic $4-6+3=1$, and hence it is a pseudotriangulation of the projective plane. The pseudocomplex $K(K_4, g)$ and its dual cell complex (whose 1-skeleton is K_4) are shown in Fig. 3 (identify diametrically opposite points on the boundary).

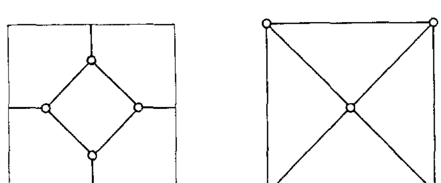

FIGURE 3

We proceed with basic combinatorial properties of pseudocomplexes which are of the form $K(G, g)$. Let K be such a complex. It is pure by construction. It is connected if and only if G is connected . More precisely, each connected component of G gives rise to a connected component of K which is, moreover, strongly connected. Let n denote the dimension of K. By construction, each $(n-1)$-simplex is contained in at most two n-simplices of K. If A is any simplex of K, then by Proposition 3.2 the star and even the open star of A are strongly connected since the subgraph which corresponds to these subpseudocomplexes is connected. Thus $K(G, g)$ is a relative n-cycle. It is important that each relative n-cycle can be realized in this way. This is established by the next theorem. Before we state it, we need a construction which gives for a relative n-cycle K a simplicial scheme g on the dual graph G of K. Define g as follows. Each arc e of $S(G)$ corresponds to a unique $(n-1)$-face of an n-simplex A of K. Let face(e) be this $(n-1)$-simplex, and let vert(e) be the vertex of K which is opposite to face(e) in A. If $e \in \mathrm{st}(i(f), G)$, $e' \in \mathrm{st}(t(f), G)$, and $\mathrm{vert}(e) = \mathrm{vert}(e')$, then define $g(f)\, e := e'$. The obtained collection of maps $\{g(f); f \in S(G),\ g(f)\colon \mathrm{st}(i(f), G) \to \mathrm{st}(t(f), G)\}$ is clearly a simplicial scheme on G. The pair (G, g) obtained from K by the above construction is denoted by $\Gamma(K)$.

3.8. Theorem. *Let G be an $(n+1)$-regular graph, and let g be a simplicial scheme on G. The pseudocomplex $K(G, g)$ is a relative n-cycle, and $\Gamma(K(G, g)) \approx (G, g)$.*

(b) *Let M be a relative n-cycle. Then $K(\Gamma(M)) \approx M$.*

Proof. For (a) we have shown above everything except that $\Gamma(K(G, g)) = (G, g)$. But this is clear by the constructions.

To verify (b), note that $K(\Gamma(M))$ is obtained as follows. Let $B_1, B_2, \ldots, B_p$ be the n-simplices of M. Take a set of disjoint n-simplices $\{A_j;\ 1 \leqq j \leqq p\}$, and for each j take a simplicial isomorphism $f_j\colon A_j \to B_j$. If two n-simplices of M, B_j and B_k, have a common $(n-1)$-face F, then identify the corresponding $(n-1)$-faces $f_j^{-1}(F)$ and $f_k^{-1}(F)$ of A_j and A_k, respectively. As the identification map take $f_j^{-1} f_k$ restricted to $f_k^{-1}(F)$. After doing all such identifications we obtain a pseudocomplex which is isomorphic to $K(\Gamma(M))$.

Define a simplicial map $h\colon K \to M$, $K = K(\Gamma(M))$, such that the following diagram is commutative

$$\begin{tikzcd}
\bigcup_{1 \leqq j \leqq p} A_j \arrow[rr, "\bigcup_{1 \leqq j \leqq p} f_j"] \arrow[dr, "q"'] & & M \\
& K \arrow[ur, "h"'] &
\end{tikzcd}$$

where q is the natural projection onto K (K is a quotient pseudocomplex of $U\{A_j; 1 \leqq j \leqq p\}$). If h exists, it will be unique. Now we prove that h is well defined. To see this, it suffices to establish that if $q(A) = q(B)$, A a face of A_j, B face of A_k, then $f_j(A) = f_k(B)$. If $q(A) = q(B)$ then there is a sequence $j = i_1, i_2, ..., i_s = k$ such that B_{i_r} and $B_{i_{r+1}}$ have a common $(n-1)$-face which contains $f_j(A)$, $r = 1, 2, ..., s-1$. Since the identification map between A_{i_r} and $A_{i_{r+1}}$ is $f_{i_{r+1}}^{-1} f_{i_r}$, it follows that $q(f_{i_r}^{-1}(f_j(A))) = q(f_{i_{r+1}}^{-1}(f_j(A)))$. By transitivity we conclude that $q(f_j^{-1}(f_j(A))) = q(f_k^{-1}(f_j(A)))$. Since this is equal to $q(A) = q(B)$, it follows that $f_k^{-1}(f_j(A)) = B$, and $f_j(A) = f_k(B)$.

Now h is a simplicial map which is onto by the construction. To prove that it is also 1–1 (and thus establishing the theorem) it suffices to see that if a simplex $A \in M$ is a face of both B_j and B_k then $qf_j^{-1}(A) = qf_k^{-1}(A)$. By the assumptions of the theorem, the open star of A is strongly connected. Therefore there exists a sequence $j = i_1, i_2, ..., i_s = k$ such that for $r = 1, 2, ..., s-1$, B_{i_r} and $B_{i_{r+1}}$ have a common $(n-1)$-face which contains A. As above, we see that $qf_{i_r}^{-1}(A) = qf_{i_{r+1}}^{-1}(A)$, and consequently $qf_j^{-1}(A) = qf_k^{-1}(B)$. ∎

The next property which is of interest is the orientability of $K(G, g)$. Let K be a relative n-cycle. An orientation of K is a choice of orientations of its n-simplices which are coherent w.r.t. all $(n-1)$-simplices of K. This is equivalent to having an orientation (in the usual sense) of the (open) manifold K minus the $(n-2)$-skeleton. K is orientable if it admits an orientation. In this case, each component of K admits exactly two different orientations. Recall that the orientation of an n-simplex can be represented as a sequence of its vertices, and two sequences represent the same orientation if they differ by an even permutation.

Let $W = f_1 f_2 \cdots f_d$ be a closed walk in G. Then $g(W)$ can be viewed in an obvious way as a permutation of arcs of $\mathrm{st}(i(f_1), G)$. Define $\mathrm{par}(W, g)$ to be equal to zero if $g(W)$ is an even permutation, and equal to 1 otherwise. Note that $\mathrm{par}(W, g)$ is well defined and invariant with respect to cyclic permutations of W.

3.9. Theorem. *$K(G, g)$ is orientable if and only if for each closed walk W, $par(W, g)$ is equal to the parity of the number of proper edges of W.*

Proof. We may suppose that G (and consequently $K = K(G, g)$) is connected. Let A be an n-simplex of K, and choose an orientation of A. If B is an n-simplex, which has a common $(n-1)$-face with A, then the orientation of A induces a coherent orientation in B. If W is a walk of n-simplices which starts at A, the orientation of A induces an orientation of the terminal simplex of the walk. K is orientable if and only if each closed walk starting at A induces at A the original orientation.

Let A and B be as above, and let f be the arc of G which corresponds to the common $(n-1)$-face of A and B. Let v be the vertex of G corresponding to A, and let $e_1, e_2, \ldots, e_n$ be the arcs of $\mathrm{st}(v, G)$ different from f. Since for $j = 1, 2, \ldots, n$ vertex $\mathrm{vert}(e_j)$ is equal to $\mathrm{vert}(g(f)\, e_j)$, the orientations represented by the sequences $(\mathrm{vert}(f), \mathrm{vert}(e_1), \ldots, \mathrm{vert}(e_n))$ and $(\mathrm{vert}(g(f) f), \mathrm{vert}(g(f)\, e_1), \ldots, \mathrm{vert}(g(f)\, e_n))$ are not coherent. By extending such a conclusion along a closed walk W without loops we see that the induced orientation of A along W is represented by the sequence $(\mathrm{vert}(g(W) f), \mathrm{vert}(g(W)\, e_1), \ldots, \mathrm{vert}(g(W)\, e_n))$ if W is of even length. If W is of odd length then the induced orientation is opposite to the one represented by the above sequence. Since the orientations represented by $(\mathrm{vert}(f), \mathrm{vert}(e_1), \ldots, \mathrm{vert}(e_n))$ and $(\mathrm{vert}(g(W) f), \mathrm{vert}(g(W)\, e_1), \ldots, \mathrm{vert}(g(W)\, e_n))$ are the same if and only if $\mathrm{par}(W, g) = 0$, the theorem follows immediately. ∎

We call the reader's attention to the work of Ferri, Gagliardi, *et al.* [1, 3, 4, 5, 6, 17]. They use edge-colored graphs to describe pseudocomplexes. Their construction is just a special case of ours. It coincides with ours when the simplicial scheme is induced by an edge-coloring. The same construction is also applied by Lins [11] (only in dimension two) and Vince [18]. It is shown [4, 11, 18] that $K(G, g)$ in the case of edge-colored graphs is orientable if and only if G (minus the half edges) is bipartite. This is also immediate by Theorem 3.9 since for edge-colored graphs $g(W)$ is the identity on every closed walk W.

At the end of this section we consider the join. If K and L are pseudocomplexes then their *join*, denoted by $K * L$, is the pseudocomplex which has simplices $A * B$, where $A \in K$, $B \in L$, and the join $A * B$ of two simplices is a simplex of dimension $\dim(A) + \dim(B) + 1$ (it has the vertex set $V(A) \cup V(B)$). Note also that the empty set is a simplex in pseudocomplexes. The incidence structure in $K * L$ is defined by $A * B \leqq A' * B'$ iff $A \leqq A'$ and $B \leqq B'$. The corresponding notion for graphs is the Cartesian product. If G and H are graphs, then their *Cartesian product* $G \times H$ has vertex set $V(G) \times V(H)$, and arc-set $(S(G) \times V(H)) \cup (V(G) \times S(H))$ where for $(e, u) \in S(G) \times V(H)$, $r((e, u)) := (r(e), u)$, $i((e, u)) := (i(e), u)$, and similarly for the arcs from $V(G) \times S(H)$. Let g and h be simplicial schemes on G and on H, respectively. For each arc $(e, u) \in S(G) \times V(H)$ and for each arc $(v, f) \in V(G) \times S(H)$ we define

$$(g \times h)((e, u))(e', u) := (g(e)\, e', u), \qquad e' \in \mathrm{st}(i(e), G),$$

$$(g \times h)((e, u))(i(e), f') := (t(e), f'), \qquad f' \in \mathrm{st}(u, H),$$

$$(g \times h)((v, f))(e', i(f)) := (e', t(f)), \qquad e' \in \mathrm{st}(v, G),$$

and

$$(g \times h)((v, f))(v, f') := (v, h(f) f'), \qquad f' \in \mathrm{st}(i(f), H).$$

Then for each arc E of $S(G \times H)$, $(g \times h)(E)$ is a bijection of $\mathrm{st}(i(E), G \times H)$ onto $\mathrm{st}(t(E), G \times H)$. In the proof of the following theorem we show that $g \times h$ is a simplicial scheme on $G \times H$.

3.10. THEOREM. *Let G and H be regular graphs with simplicial schemes g and h, respectively. Then*

$$K(G \times H, g \times h) \approx K(G, g) * K(H, h).$$

Proof. First of all we prove that $g \times h$ is a simplicial scheme. It clearly satisfies (SS1) and (SS2). To further verify the property (SS3), let W be a closed walk in $G \times H$ which starts at vertex (v, u), and let E be an arc of $\mathrm{st}((v, u), G \times H)$ which avoids W. Without loss of generality assume that $E = (e, u) \in S(G) \times V(H)$. Denote by P the closed walk in G which is the projection of W. Since E avoids W, it follows that e avoids P, and hence $g(P)\, e = e$. A short calculation $(g \times h)(W)\, E = (g \times h)(W)(e, u) = (g(P)\, e, u) = (e, u) = E$ verifies (SS3).

It looks somehow clear that $K(G \times H, g \times h)$ is isomorphic to $K(G, g) * K(H, h)$ but the proof requires some accurate considerations. We shall determine $\Gamma(K(G, g) * K(H, h))$. By showing that this is isomorphic to the pair $(G \times H, g \times h)$, Theorem 3.8 will yield the required result. First we verify that the dual graph of $K := K(G, g) * K(H, h)$ is isomorphic to $G \times H$. Vertex $(v, u) \in V(G \times H)$ corresponds to top-dimensional simplex $A_v * B_u$ of K where v represents A_v in the dual of $K(G, g)$, and u represents B_u in the dual of $K(H, h)$. Arc $(e, u) \in \mathrm{st}((v, u), G \times H)$ corresponds to the codimension-one simplex $\mathrm{face}(e) * B_u$ of $A_v * B_u$, while $(v, f) \in \mathrm{st}((v, u), G \times H)$ corresponds to $A_v * \mathrm{face}(f)$.

Let z be a simplicial scheme such that $\Gamma(K) = (G \times H, z)$. Because of symmetry we show only that z and $g \times h$ agree on the arcs of $S(G) \times V(H)$. Let $(e, u) \in S(G) \times V(H)$, and let $(e', u) \in S(G) \times V(H)$ and $(i(e), f) \in V(G) \times S(H)$ be any arcs of $\mathrm{st}((i(e), u), G \times H)$. If $(e', u) = (e, u)$ then by (SS1), $(g \times h)((e, u))(e', u) = r((e, u)) = z((e, u))(e', u)$. Otherwise, let y be the vertex of $A_{i(e)} * B_u$ which is opposite $\mathrm{face}((e', u))$. Note that y is the vertex of $A_{i(e)}$ which is opposite $\mathrm{face}(e')$. In $A_{t(e)}$, $\mathrm{face}(g(e)\, e')$ is opposite y. Thus $\mathrm{face}((g(e)\, e', u))$ is opposite y in $A_{t(e)} * B_u$. In other words this means that $z((e, u))(e', u) = (g(e)\, e', u)$ which is equal to $(g \times h)((e, u))(e', u)$. Similarly we show that $z((e, u))(i(e), f) = (t(e), f) = (g \times h)((e, u))(i(e), f)$. This completes the proof. ∎

4. s-MAPS AND SIMPLICIAL MAPS

From a categorical point of view we consider the category GPH_n^s of n-regular graphs (objects of GPH_n) together with simplicial schemes. A *morphism* (or a *map*) between objects (G, g) and (H, h) of GPH_n^s is an s-map $p: G \to H$ such that for each arc $e \in S(G)$ with initial vertex v and terminal vertex u the following diagram commutes

$$\begin{array}{ccc} \mathrm{st}(v, G) & \xrightarrow{g(e)} & \mathrm{st}(u, G) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \mathrm{st}(p(v), H) & \xrightarrow{h(p(e))} & \mathrm{st}(p(u), H) \end{array} \tag{4.1}$$

A morphism $p: G \to H$ can be uniquely extended to a map between the simplicial schemes. Therefore we also write $p: (G, g) \to (H, h)$.

4.1. LEMMA. *Let $p: G \to H$ be an s-map, and let h be a simplicial scheme on H. Then there exists a uniquely determined simplicial scheme g on G such that p is a morphism of (G, g) into (H, h).*

Proof. The map p restricted to the star of a vertex is bijective, and hence for each $e \in S(G)$ the diagram (4.1) yields an exactly determined bijective map $g(e)$: $\mathrm{st}(i(e), G) \to \mathrm{st}(t(e), G)$ such that the diagram commutes. It is clear that the collection of maps $\{g(e); e \in S(G)\}$ is a presimplicial scheme. The property (SS3) is also immediate. Let $W = f_1 f_2 \ldots f_d$ be a closed walk in G and let e be an arc which avoids W. From the commutativity of the following diagram:

$$\begin{array}{ccccccccc} \mathrm{st}(v_1, G) & \xrightarrow{g(f_1)} & \mathrm{st}(v_2, G) & \xrightarrow{g(f_2)} \cdots \longrightarrow & \mathrm{st}(v_d, G) & \xrightarrow{g(f_d)} & \mathrm{st}(v_1, G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{st}(u_1, H) & \xrightarrow{hp(f_1)} & \mathrm{st}(u_2, H) & \xrightarrow{hp(f_2)} \cdots \longrightarrow & \mathrm{st}(u_d, H) & \xrightarrow{hp(f_d)} & \mathrm{st}(u_1, H) \end{array}$$

where $v_j := i(f_j)$, $u_j := p(v_j)$, $j = 1, 2, \ldots, d$, it follows that the arc $p(e)$ avoids the walk $p(W) = p(f_1)\, p(f_2) \cdots p(f_d)$. Since h is a simplicial scheme, $h(p(W))\, p(e) = p(e)$. By the commutativity of the diagram and since $p \mid \mathrm{st}(v_1)$ is 1–1, we conclude that $g(W)\, e = e$. The proof is completed. ∎

If a simplicial scheme g on G is induced by an s-map $p: G \to H$ and a simplicial scheme h on H, we say that g is the *lift* of h into G. We proceed with a theorem which indicates a deep connection between s-maps and nondegenerate simplicial maps.

4.2. THEOREM. *The assignment K of a pair (G, g) from GPH_{n+1}^s to a relative n-cycle $K(G, g)$ is an isomorphism from GPH_{n+1}^s to the category of relative n-cycles and nondegenerate simplicial maps.*

Proof. To each morphism $p: (G, g) \to (H, h)$ we assign a uniquely determined nondegenerate simplicial map $K(p)\colon K(G, g) \to K(H, h)$. $K(p)$ maps n-simplex A_v, $v \in V(G)$, to the n-simplex $B_{p(v)}$ of $K(H, h)$ which corresponds to the vertex $p(v) \in V(H)$. Similarly, for each $e \in S(G)$, face(e) is mapped to face$(p(e))$, and consequently vert(e) is mapped to vert$(p(e))$. This in turn implies that $K(p)$ is uniquely defined on 0-simplices. However, it is not clear that $K(p)$ is well defined. Let $e, f \in S(G)$, and suppose that vert$(e) =$ vert(f). All we have to show is that vert$(p(e)) =$ vert$(p(f))$. By Proposition 3.1 there is a walk W in G from $i(e)$ to $i(f)$ such that e avoids W and $f = g(W)\, e$. As in the proof of Lemma 4.1 we see that $p(e)$ avoids the walk $p(W)$ and that $h(p(W))\, p(e) = p(f)$. This implies that vert$(p(e)) =$ vert$(p(f))$, thus $K(p)$ is well defined.

We leave the relations $K(id_{(G,g)}) = \mathrm{id}_{K(G,g)}$ and $K(qp) = K(q)\, K(p)$ to the reader. These imply that K is a covariant functor.

Let $q\colon K(G, g) \to K(H, h)$ be a nondegenerate map. For convenience suppose that G and H are dual graphs of $K(G, g)$ and $K(H, h)$, respectively, i.e., $V(G)$ is the set of n-simplices of $K(G, g)$, and the arcs of G are the pairs (A, B), A n-simplex, B $(n-1)$-face of A, and similarly for H. Define a graph map $p\colon G \to H$ as follows. For $A \in V(G)$, let $p(A) := q(A) \in V(H)$, and for $(A, B) \in S(G)$, let $p((A, B)) := (q(A), q(B))$. It is clear that p is an s-map. A proof that p is a morphism of (G, g) to (H, h) is left to the reader. By the definition of p, it is obvious that $q = K(p)$. Together with Theorem 3.8 this implies that K is an isomorphism of categories. ▌

Let K and L be pure pseudocomplexes of dimension n. A nondegenerate map $f\colon K \to L$ is *nonsingular* if any two n-simplices having a common $(n-1)$-face are mapped under f onto different n-simplices in L. In Fig. 4 we give an example of nonsingular and singular behaviour of a simplicial map.

4.3. THEOREM. *Let $p\colon (G, g) \to (H, h)$ be a morphism in GPH^s_{n+1}. If H is connected, then the following statements are equivalent:*

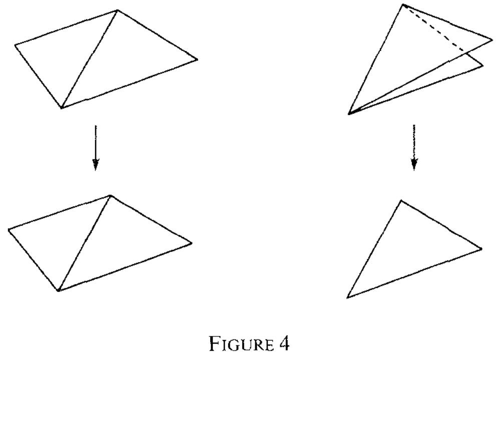

FIGURE 4

(1) *$K(p)$: $K(G, g) \to K(H, h)$ is nonsingular;*

(2) *every edge of G is nonsingular* w.r.t. p;

(3) *for each $e \in S(G)$, if $e \neq r(e)$ then $p(e) \neq pr(e)$.*

Proof. The equivalence of (2) and (3) is clear. Thus we prove only that the statements (1) and (3) are equivalent.

(1) $\Rightarrow$ (3) Suppose that $K(p)$ is nonsingular, and let $e \in S(G)$ be an arc of G such that $e \neq r(e)$. Then $i(e) \neq t(e)$. The n-simplices of $K(G, g)$ corresponding to $i(e)$ and $t(e)$ are adjacent, and since $K(p)$ is nonsingular they are mapped onto different n-simplices in $K(H, h)$. This means that $p(i(e)) \neq p(t(e))$. Consequently, $ip(e) \neq ipr(e)$, and hence $p(e) \neq pr(e)$.

(3) $\Rightarrow$ (1) Suppose that $K(G, g)$ has two adjacent n-simplices, A and B, which are mapped under $K(p)$ onto the same n-simplex of $K(H, h)$. As above we see for the corresponding arc $e \in S(G)$ that $pi(e) = pt(e)$, whereas $r(e) \neq e$ since $i(e)$ and $t(e)$ are different vertices corresponding to A and B. From $pi(e) = pt(e)$ one obtains that either e is a singular arc or $p(e)$ is a half-edge. In both cases $p(e) = pr(e)$. ∎

A nonsingular mapping $f: K \to L$ is a *branched covering* if $f(\text{bd}(K)) = \text{bd}(L)$. By Theorem 4.3 it is clear that branched coverings can be described by s-maps which are nonsingular and map the half-edges of the above graph onto the half-edges of the downstairs graph. Such graph maps are called graph covering projections [7, 8].

4.4. THEOREM. *Let p: $(G, g) \to (H, h)$ be a morphism in GPH^s_{n+1}. If H is connected then the mapping $K(p)$: $K(G, g) \to K(H, h)$ is a branched covering if and only if p is a graph covering projection $G \to H$.*

Theorem 4.3 is an important tool when considering nonsingular maps and Theorem 4.4 is very useful by examination of branched coverings between relative cycles. Applications of this can be found in [12, 14 ,15].

Now we consider the vertex-colorability of pseudocomplexes. An *m-coloring* of a pseudocomplex K is a partition of the vertex set $V(K)$ into m color classes $V_0, V_1, \ldots, V_{m-1}$ such that no two vertices in the same color class belong to the same simplex of K or, equivalently, are not joined by an edge of K.

4.5. THEOREM. *Let K be a relative n-cycle. Then K admits an $(n+1)$-coloring if and only if there is a map p: $\Gamma(K) \to (W_{n+1}, w_{n+1})$* where W_{n+1} is defined in Example 3.6.

Proof. ($\Rightarrow$) Let $\Gamma(K) = (G, g)$ and suppose that $K = K(G, g)$ has an $(n+1)$-coloring. Define a map p: $(G, g) \to (W_{n+1}, w_{n+1})$ as follows. It maps the vertices of G onto the vertex of W_{n+1}. Label the arcs of W_{n+1} by $f_0, f_1, ..., f_n$. For $e \in S(G)$, let $p(e) := f_j$ if vert(e) is in the jth color class of the $(n+1)$-coloring. Note that vert(e) has the same color as vert($r(e)$). Thus p is graph map. Since on a given n-simplex of K every color is used exactly once, p is also an s-map. It is clear that the simplicial scheme g is the lift of w_{n+1} w.r.t. p since for each arc $f \in \text{st}(i(e), G)$, f and $g(e)f$ are opposite to the same vertex (or opposite to vertices of the same color if $e=f$), and thus they project to the same arc of W_{n+1}.

($\Leftarrow$) If $p: (G, g) \to (W_{n+1}, w_{n+1})$ is a map in GPH^s_{n+1}, the following defines an $(n+1)$-coloring of $K(G, g)$. To a vertex α of K give the color j if $\alpha = \text{vert}(e)$ for an arc $e \in S(G)$ such that $p(e) = f_j$. ∎

If $p: (G, g) \to (W_{n+1}, w_{n+1})$ is a morphism then the simplicial scheme g is induced by an edge-coloring of G which is the lift of the efge-coloring of W_{n+1}. In this case $K(G, g)$ is orientable if and only if G is bipartite as it was shown in Section 3. This in turn implies the following result.

4.6. Corollary. *$K(G, g)$ is orientable and has an $(n+1)$-coloring if and only if (G, g) maps nonsingularly to (H_{n+1}, h_{n+1}) where the pair (H_{n+1}, h_{n+1}) is defined in Example* 3.5.

Let $K(G, g)$ be an orientable n-cycle without boundary. If $K(G, g)$ has an $(n+1)$-coloring, then by Theorem 4.3 and Corollary 4.6., G is a covering graph over H_{n+1} and the simplicial scheme on G is exactly determined as the lift of h_{n+1}. This indicates the possible use of graph coverings in the theory of colorability of pseudocomplexes.

Finally we give an example which shows how simplicial schemes can be easily used to prove a combinatorial property of relative cycles. Suppose that a relative n-cycle K has the following properties:

(1) K is simply connected,

(2) every simplex of K has a simply connected neighborhood, and

(3) every interior $(n-2)$-simplex is contained in an even number of n-simplices.

Then the dual graph of K is bipartite.

We sketch a proof of the above result using simplicial schemes. It is as follows. Let $K = K(G, g)$. The properties of the Kronecker cover $G \otimes K_2$ (the tensor product with K_2) of the graph G and the lift $\tilde{g}$ of g to $G \otimes K_2$ can be used to show that $K(G \otimes K_2, \tilde{g})$ is a covering space over K. If K is simply connected this covering cannot be connected, thus the graph $G \otimes K_2$ is not connected. This in turn implies that G is bipartite.

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